ON THE NUMBER OF HOMOGENEOUS MODELS OF A GIVEN POWER*

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ABSTRACT

It is shown that, For each complete theory T, the nomber $h_T(\mathfrak{m})$ of homogeneous models of T of power $\mathfrak{m}$ is a non-increasing function of uncountabel cardinals $\mathfrak{m}$ Moreover, if $h_T(\aleph_0) \leqq \aleph_0$, then the function h_T is also non-increasing $\aleph_0$ to $\aleph_1$.

A model $\mathfrak{A}$ is said to be homogeneous if, roughly speaking, any isomorphism between two small elementary submodels of $\mathfrak{A}$ can be extended to an automorphism of $\mathfrak{A}$. The notion is due to Morley and Vaught [7], and is based on Jónsson [2] (a precise definition is stated below). Consider a complete theory T in a countable first order logic with identity. For each infinite cardinal cardinal $\aleph_\alpha$, let $h_T(\aleph_\alpha)$ be the number of (non-isomorphic) homogeneous models of T of power $\aleph_\alpha$. We shall prove the following two theorems.

THEOREM A. *If* $h_T(\aleph_0) \leqq \aleph_0$, *then for all* $\mathfrak{m} > \aleph_0$, $h_T(\mathfrak{m}) \leqq h_T(\aleph_0)$.

THEOREM B. *Assume the GCH (generalized continuum hypothesis). If* $\aleph_1 \leqq \mathfrak{m} \leqq \mathfrak{n}$, *then* $h_T(\mathfrak{m}) \geqq h_T(\mathfrak{n})$.

COROLLARY C (*GCH*). *The function* h_T *is non-increasing except for the single possibility*

$$h_T(\aleph_0) = \aleph_1,\ h_T(\aleph_1) = \aleph_2.$$

It is shown in [7] that if T has any infinite models then $h_T(\aleph_0) > 0$, and assuming the GCH, $h_T(\mathfrak{m}) > 0$ for all infinite $\mathfrak{m}$. Some other results on homogeneous models are given in [4]. For results on the total number of models of T of a given power see [5], [6], and [9].

In this paper we shall use $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ to denote models with universe sets A, B, C. $T(\mathfrak{A})$ stands for the complete theory of $\mathfrak{A}$, that is, the set of all sentences of our

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given logic L which are true of $\mathfrak{A}$. If a is an α-termed sequence of elements of A, then $(\mathfrak{A}, a)$ is a model for the language L with α additional individual constants; in this case, $T(\mathfrak{A}, a)$ is called the *type* of the sequence a in $\mathfrak{A}$. We shall be especially interested in types of finite sequences of elements, and we let $S(\mathfrak{A})$ be the set of all types of finite sequences o elements of A,

$$S(\mathfrak{A}) = \{T(\mathfrak{A}, a) : a \in \bigcup_{n<\omega} A^n\}.$$

The symbol $\mathfrak{A} \prec \mathfrak{B}$ means that $\mathfrak{B}$ is an elementary extension of $\mathfrak{A}$ (or $\mathfrak{A}$ is an elementary submodel of $\mathfrak{B}$). We assume the reader knows the definition and basic properties of $\mathfrak{A} \prec \mathfrak{B}$, which are given by Tarski and Vaught in [8]. Note that

$$\mathfrak{A} \prec \mathfrak{B} \qquad \text{implies } S(\mathfrak{A}) \subseteq S(\mathfrak{B}),$$

$$S(\mathfrak{A}) \subseteq S(\mathfrak{B}) \text{ implies } T(\mathfrak{A}) = T(\mathfrak{B}).$$

DEFINITION [7].(2) Let $\mathfrak{A}$ be a model of infinite power $|A| = \mathfrak{m}$.

CASE 1. $\mathfrak{m}$ is regular. $\mathfrak{A}$ is *homogeneous* if for all $\mathfrak{B}, \mathfrak{C} \prec \mathfrak{A}$ of power less than $\mathfrak{m}$, any isomorphism f on $\mathfrak{B}$ onto $\mathfrak{C}$ can be extended to an automorphism of $\mathfrak{A}$.

CASE 2. $\mathfrak{m}$ is singular. $\mathfrak{A}$ is *homogeneous* if $\mathfrak{A}$ is the union of an elementary chain

$$(*) \qquad \mathfrak{A}_0 \prec \mathfrak{A}_1 \prec \cdots \prec \mathfrak{A}_\beta \cdots$$

where each $\mathfrak{A}_\beta$ is a homogeneous model of a regular power less than $\mathfrak{m}$ such that $S(\mathfrak{A}_\beta) = S(\mathfrak{A})$.

The following lemma from [7] puts the definition of homogeneous model in a more convenient form.

LEMMA 1. *Let $\mathfrak{m}$ be an infinite regular cardinal. The following is a necessary and sufficient condition for a model $\mathfrak{A}$ of power $\mathfrak{m}$ to be homogeneous:*

For all ordinals $\alpha < \mathfrak{m}$, all α-termed sequences $a, b \in A^\alpha$, and all $c \in A$, if

$$T(\mathfrak{A}, a) = T(\mathfrak{A}, b)$$

then there exists $d \in A$ such that

$$T(\mathfrak{A}, a, c) = T(\mathfrak{A}, b, d).$$

We shall now prove a series of lemmas from which Theorems A–C will follow at once.

LEMMA 2. *If $\mathfrak{A}$ is homogeneous, $S(\mathfrak{B}) \subseteq S(\mathfrak{A})$, and $|B| \leqq |A|$, then $\mathfrak{B}$ is isomorphic to an elementary submodel of $\mathfrak{A}$.*

(2) This definition differs from the one in [7] when $\mathfrak{m}$ is singular.

Proof. First let the cardinal $\mathfrak{m}$ of A be regular. By transfinite induction on $\mathfrak{N}$ it can be shown that for each cardinal $\mathfrak{n} \leqq \mathfrak{m}$, for all $b \in B^{\mathfrak{n}}$ there exists $a \in A^{\mathfrak{n}}$ such that

$$T(\mathfrak{A}, a) = T(\mathfrak{B}, b).$$

The desired result follows when we take $b \in B^{\mathfrak{m}}$ whose range is B.

Now let $\mathfrak{m}$ be singular and represent $\mathfrak{A}$ as the union of a chain (*). Let $\mathfrak{B}$ be the union of an elementary chain

$$\mathfrak{B}_0 \prec \mathfrak{B}_1 \prec \cdots \prec \mathfrak{B}_\beta \prec \cdots$$

where each $\mathfrak{B}_\beta$ has power at most the power of $\mathfrak{A}_\beta$. Then for each β there is an isomorphism f_β of $\mathfrak{B}_\beta$ onto an elementary submodel of $\mathfrak{A}_\beta$. We may assume that the sequence (*) has been thinned out so that for each β,

$$|A_\beta| > |\textstyle\bigcup_{\gamma<\alpha} A_\gamma|.$$

Then using the fact that the $\mathfrak{A}_\beta$ are homogeneous, we may choose the isomorphisms f_β so that

$$\textstyle\bigcup_{\gamma<\beta} f_\gamma \subseteq f_\beta.$$

The union of all the f_β is the desired isomorphism from $\mathfrak{B}$ to an elementary submodel of $\mathfrak{A}$.

LEMMA 3. *If $\mathfrak{A}, \mathfrak{B}$ are homogeneous, $|A| = |B|$, and $S(\mathfrak{A}) = S(\mathfrak{B})$, then $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic.*

Proof. Similar to Lemma 2. Going back and forth between $\mathfrak{A}$ and $\mathfrak{B}$, an isomorphism can be constructed with exhausts both A and B.

LEMMA 4. *For any complete theory T and any infinite cardinal $\mathfrak{m}$,*

$$h_T(\mathfrak{m}) \leqq 2^{2^{\aleph_0}} \text{ and } h_T(\mathfrak{m}) \leqq 2^{\mathfrak{m}}.$$

Proof. There are only $2^{\aleph_0}$ complete types of finite sequences, hence only $2^{2^{\aleph_0}}$ possible sets $S(\mathfrak{A})$. Moreover, there are only $2^{\mathfrak{m}}$ models of power $\mathfrak{m}$, up to isomorphism.

LEMMA 5. *Suppose $\mathfrak{A}$ is a homogeneous model, $X \subseteq S(\mathfrak{A})$, and $|X| \leqq \aleph_0$. Then there is a countable homogeneous model $\mathfrak{B} \prec \mathfrak{A}$ such that $X \subseteq S(\mathfrak{B})$.*

Proof. Let C be a countable subset of A such that for each $x \in X$ there is an n-tuple of elements of C of type x in $\mathfrak{A}$. It suffices to find a countable homogeneous model $\mathfrak{B} \prec \mathfrak{A}$ such that $C \subseteq B$. First choose a countable $\mathfrak{B}_0 \prec \mathfrak{A}$ such that $C \subseteq B_0$. Let C_1 be a countable set such that $B_0 \subseteq C_1 \subseteq A$ and for all $a, b \in B_0^n$ and $c \in B_0$, if

$$T(\mathfrak{A}, a) = T(\mathfrak{A}, b)$$

then there is a $d \in C_1$ such that

$$T(\mathfrak{A}, a, c) = T(\mathfrak{A}, b, d).$$

Choose a countable $\mathfrak{B}_1 \prec \mathfrak{A}$ such that $C_1 \subseteq B_1$. Continuing the process we choose countable models

$$\mathfrak{B}_0 \prec \mathfrak{B}_1 \prec \mathfrak{B}_2 \prec \cdots$$

and let $\mathfrak{B}$ be the union of the chain. Then $\mathfrak{B}$ is homogeneous and $X \subseteq S(\mathfrak{B})$.

Theorem A now follows easily. We note that if $\mathfrak{B}$ is countable then $S(\mathfrak{B})$ is countable. Hence if the theory T has a homogeneous model $\mathfrak{A}$ such that $S(\mathfrak{A})$ is uncountable, then by Lemma 5 we have $h_T(\aleph_0) > \aleph_0$. Suppose, on the other hand, that $S(\mathfrak{A})$ is countable for every homogeneous model $\mathfrak{A}$ of T. Then for each homogeneous model $\mathfrak{A}$ of T there is a countable homogeneous model $\mathfrak{B}$ with $S(\mathfrak{B}) = S(\mathfrak{A})$ (by Lemma 5). Hence by Lemma 3, $h_T(\mathfrak{m}) \leqq h_T(\aleph_0)$ for every $\mathfrak{m} \geqq \aleph_0$.

LEMMA 6 (GCH). *Let A be a homogeneous model of power $\mathfrak{n}$ and let $\aleph_1 \leqq \mathfrak{p} \leqq \mathfrak{n}$. Then there is a homogeneous model $\mathfrak{B} \prec \mathfrak{A}$ of power $\mathfrak{p}$ such that $S(\mathfrak{B}) = S(\mathfrak{A})$.*

Proof. From the definition we may clearly assume that n is regular. It suffices to prove that for each $C \subseteq A$ and each regular cardinal $\mathfrak{m}$ with $\aleph_1 \leqq |C| \leqq \mathfrak{m} \leqq \mathfrak{n}$, there is a homogeneous $\mathfrak{B} \prec \mathfrak{A}$ of power $\mathfrak{m}$ with $C \subseteq B$.

Choose a $\mathfrak{B}_0 \prec \mathfrak{A}$ of power $\mathfrak{m}$ with $C \subseteq B_0$. By the GCH, there are $\mathfrak{m}$ sequences $a \in \bigcup_{\alpha<\mathfrak{m}} B^\alpha$. Hence there is a set C_1 of power $\mathfrak{m}$ such that $B_0 \subseteq C_1 \subseteq A$ and for all sequences a, b in B_0 of length $\alpha < \mathfrak{m}$ and all $c \in B_0$, if

$$T(\mathfrak{A}, a) = T(\mathfrak{A}, b)$$

then there exists $d \in C_1$ such that

$$T(\mathfrak{A}, a, c) = T(\mathfrak{A}, b, d).$$

We choose $\mathfrak{B}_1 \prec \mathfrak{A}$ of power $\mathfrak{m}$ containing C_1, and continue the process $\mathfrak{m}$ times to form an elementary chain

$$\mathfrak{B}_0 \prec \mathfrak{B}_1 \prec \cdots \prec \mathfrak{B}_\alpha \prec \cdots$$

of length $\mathfrak{m}$ with each B_α of power $\mathfrak{m}$. The union of the chain will be homogeneous of power $\mathfrak{m}$.

If $\mathfrak{p}$ is regular, take $\mathfrak{p} = \mathfrak{m}$ and take C to contain sequences of each type in $S(\mathfrak{A})$; note that $|S(\mathfrak{A})| \leqq \aleph_1 \leqq \mathfrak{p}$. For $\mathfrak{p}$ singular we iterate the above result to get an elementary chain of homogeneous models $\mathfrak{A}_\mathfrak{m}$ of power $\mathfrak{m}$ for each regular $\mathfrak{m}$, $\aleph_1 \leqq \mathfrak{m} < \mathfrak{p}$, with $S(\mathfrak{A}_{\mathfrak{m}}) = S(\mathfrak{A})$. Then the union is homogeneous of power $\mathfrak{p}$. This completes the proof.

Theorem B is now immediate. By Lemma 6, for each homogeneous model $\mathfrak{A}$ of T of power $\mathfrak{n}$ there is a homogeneous model $\mathfrak{B}$ of power $\mathfrak{m}$ with $S(\mathfrak{B}) = S(\mathfrak{A})$. By Lemma 3 we have $h_T(\mathfrak{m}) \geqq h_T(\mathfrak{n})$. Corollary C follows from A, B, and Lemma 4.

Our lemmas give more general versions of Theorems A, B, and Corollary C. Let $L_{\omega_1\omega}$ be the logic which is like first order logic but has countably infinite conjunctions, and let $L_{\omega_1\omega_1}$ be the logic which has countably infinite conjunctions and quantifiers over countable sequences of variables (cf. [3]). If T is a countable set of sentences of $L_{\omega_1\omega}$, then the results A, B and C all hold for T. Theorem B still holds if T is a set of sentences of $L_{\omega_1\omega_1}$ and $\mathfrak{m}, \mathfrak{n}$ are not cofinal with ω. Moreover, both of the above remarks still hold true if the sentences of T have countably many second order existential quantifiers at the beginning. The point is that in Lemmas 5 and 6, if $\mathfrak{A}$ is a model of T then $\mathfrak{B}$ can be taken to be a model of T.

Instead of giving direct proofs of Lemmas 5 and 6, we could have deduced them as special cases of Löwenheim-Skolem theorems for the appropriate infinitary logics in [3].

We conclude with some problems and examples. Assume the GCH from now on.

By Theorem B, each complete first order theory T has a least cardinal $\mathfrak{m}_1(T)$ such that for all $\mathfrak{m}$, $\mathfrak{n} \geqq \mathfrak{m}_1(T)$, $h_T(\mathfrak{m}) = h_T(\mathfrak{n})$. Since there are only $2^{\aleph_0}$ T's, there is a least $\mathfrak{m}_1$ such that $\mathfrak{m}_1 \geqq \mathfrak{m}_1(T)$ for all T. What is the cardinal $\mathfrak{m}_1$? All we know so far is that $\mathfrak{m}_1 \geqq \aleph_2$ (from the examples below). There is also a least cardinal $\mathfrak{m}_2$ such that for every homogeneous model $\mathfrak{A}$ of power $\geqq \mathfrak{m}_2$ and all $\mathfrak{n} \geqq \mathfrak{m}_2$, there is a homogeneous model $\mathfrak{B}$ of power $\mathfrak{n}$ with $S(\mathfrak{B}) = S(\mathfrak{A})$ (by Lemma 6). It follows from Lemmas 3 and 6 that $\mathfrak{m}_2 \geqq \mathfrak{m}_1$. The value of $\mathfrak{m}_2$ is not known, but seems easier to get at than $\mathfrak{m}_1$. The argument that the numbers $\mathfrak{m}_1, \mathfrak{m}_2$ exist works in a wide variety of situations and is due to Hanf [1]; such numbers are sometimes called Hanf numbers.

For each finite or infinite cardinal $\mathfrak{n} \leqq \aleph_1$, there is a theory T such that h_T is the constant function with value $\mathfrak{n}$. Any theory T categorical in power $\aleph_0$ has $h_T(\mathfrak{m}) = 1$ for all $\mathfrak{m}$. For $3 \leqq n < \omega$, the theories T_n of Ehrenfeucht discussed in [9], p. 318, have $h_{T_n}(\mathfrak{m}) = n$ for all $\mathfrak{m}$. It is easy to find theories T where h_T has the constant value $\aleph_0$, or $\aleph_1$. The complete theory of the model $\langle A, I, < \rangle$ where I is the set of all integers, $<$ the natural order on I, and $A - I$ is countably infinite is an example of a theory T with $h_T(\mathfrak{m}) = 2$ for all $\mathfrak{m}$.

The complete theory of the countable tree in which each element has finitely many predecessors and exactly two immediate successors is a theory T with $h_T(\aleph_0) = 3$, $h_T(\aleph_1) = 2$, $h_T(\aleph_2) = 1$. Whenever $3 \geqq a \geqq b \geqq c$ we know an example of a theory T with

$$h_T(\aleph_0) = a, \quad h_T(\aleph_1) = b, \quad h_T(\aleph_2) = c.$$

We have not yet found any example of a theory T with

$$h_T(\aleph_0) = 4, \ h_T(\aleph_1) = 1,$$

or even with

$$\aleph_0 > h_T(\aleph_0) > 3\, h_T(\aleph_1).$$

The complete theory of a model with unary relations $R_0, R_1, \cdots$ such that each non-trivial Boolean combination of the R_i is infinite is a theory T with

$$h_T(\aleph_0) = \aleph_1, \ h_T(\mathfrak{m}) = \aleph_2 \text{ for } \mathfrak{m} > \aleph_0.$$

The theory of a model $\langle A, c_0, c_1, c_2, \cdots \rangle$ where the c_i are distinct constants is a T with

$$h_T(\aleph_0) = \aleph_0, \ h_T(\aleph_1) = 1.$$

We do not know whether there is a complete theory T which does any of the following:

$$h_T(\aleph_0) = \aleph_0, \quad 1 < h_T(\aleph_1) < \aleph_0.$$

$$h_T(\aleph_0) = \aleph_1, \quad h_T(\aleph_1) \leqq \aleph_0.$$

$$h_T(\aleph_0) \geqq \aleph_0 \text{ and } h_T(\aleph_1) > h_T(\aleph_2),$$

$$h_T(\aleph_2) > h_T(\aleph_3).$$

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